Spherically symmetric random walks. II. Dimensionally dependent critical behavior

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A recently developed model of random walks on a D-dimensional hyperspherical lattice, where D is *not* restricted to integer values, is extended to include the possibility of creating and annihilating random walkers. Steady-state distributions of random walkers are obtained for all dimensions $D>0$ by solving a discrete eigenvalue problem. These distributions exhibit dimensionally dependent critical behavior as a function of the birth rate. This remarkably simple model exhibits a second-order phase transition with a universal, nontrivial critical exponent for all dimensions $D>0$. [S1063-651X(96)05706-6]

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I. INTRODUCTION

In previous papers [1–3] we analyze a class of models of D-dimensional spherically symmetric random walks, where D is *not restricted to integer values*. In this paper we extend these models to allow for the creation and annihilation of random walkers. We demonstrate that these extended models exhibit critical behavior as a function of the birth rate of walkers. The critical coefficients depend on the value of the dimension D. Universality for the critical properties of this model has been demonstrated both analytically and numerically [3].

Random walks with sources and traps are widely used to describe a variety of interesting physical phenomena such as chemical reactions [4] and diffusion in random media [5,6]. Our random-walk model on a hyperspherical lattice makes the consideration of a free random walk in the neighborhood of an entrapping boundary *in arbitrary dimensions* particularly simple. In the next paper in this series we apply these ideas to the study polymer growth in D dimensions in the vicinity of a hyperspherical adsorbing boundary [7].

The random walks in Refs. [1–3] take place on an infinite set of regions labeled by the integer n, $n=1,2,3,\ldots$. If the random walker is in *region n* at time t, then at time $t+1$ the walker must move outward to region $n+1$ with probability $P_{\rm out}(n)$ or inward to region $n-1$ with probability $P_{\rm in}(n)$, where

$$P_{\rm out}(n)+P_{\rm in}(n)=1 \tag{1.1}$$

so that probability is conserved. [We take $P_{\rm out}(1)=1$ and $P_{\rm in}(1)=0$ to enforce the requirement that a walker in the central region $(n=1)$ must move outward at the next step.] Let $C_{n,t;m}$ represent the probability that a random walker who begins in the mth region at $t=0$ will be in the nth region at time t. The probability $C_{n,t;m}$ then satisfies the difference equation

$$C_{n,t;m}=\begin{cases} P_{\rm in}(n+1)C_{n+1,t-1;m}+P_{\rm out}(n-1)C_{n-1,t-1;m} & (n\geqslant 2)\\ P_{\rm in}(2)C_{2,t-1;m} & (n=1)\end{cases} \tag{1.2}$$

and the initial condition

$$C_{n,0;m}=\delta_{n,m}. \tag{1.3}$$

To formulate a model of spherically symmetric random walks in D-dimensional space we take region n to be the volume bounded by two concentric D-dimensional hyperspherical surfaces of radii R_{n-1} and R_n. In Ref. [1] we take the probabilities of moving out or in to be in proportion to the hyperspherical surface areas bounding region n. Let $S_D(R)$ represent the surface area of a D-dimensional hypersphere

$$S_D(R)=\frac{2\pi^{D/2}}{\Gamma(D/2)}R^{D-1}.$$

Then, for $n>1$,

$$P_{\rm out}(n)=\frac{S_D(R_n)}{S_D(R_n)+S_D(R_{n-1})}=\frac{R_n^{D-1}}{R_n^{D-1}+R_{n-1}^{D-1}} \quad (1.4)$$

and

$$P_{\rm in}(n)=\frac{S_D(R_{n-1})}{S_D(R_n)+S_D(R_{n-1})}=\frac{R_{n-1}^{D-1}}{R_n^{D-1}+R_{n-1}^{D-1}}. \quad (1.5)$$

As we discussed earlier, for the special case $n=1$ we define

$$P_{\rm out}(1)=1, \quad P_{\rm in}(1)=0. \quad (1.6)$$

The choices in Eqs. (1.4)–(1.6) satisfy the requirement that probability be conserved because they obey Eq. (1.1).

For dimensions other than $D=1$ and 2, when we substitute Eqs. (1.4)–(1.6) into Eq. (1.2) and take $R_n=n$, we obtain a difference equation that cannot be solved in closed form. Thus, in Ref. [2] we proposed that the probabilities in Eqs. (1.4)–(1.6) be replaced by bilinear functions of n, which are a uniformly good approximations to $P_{\rm out}(n)$ and $P_{\rm in}(n)$ in the range $D>0$ when $R_n=n$:

$$P_{\rm out}(n)=\frac{n+D-2}{2n+D-3}, \quad P_{\rm in}(n)=\frac{n-1}{2n+D-3}. \quad (1.7)$$

Now, the difference equation initial-value problem (1.2) and (1.3) for the probabilities $C_{n,t;m}$ can be solved in closed form

$$C_{n,t;m}=\frac{(2n+D-3)\Gamma^2(D-1)\Gamma(m)}{2^{D-1}\Gamma^2(D/2)\Gamma(m+D-2)} \times\int_{-1}^{1}dx(1-x^2)^{(D-2)/2}x^t\mathscr{C}_{n-1}^{[(D-1)/2]}(x) \times\mathscr{C}_{m-1}^{[(D-1)/2]}(x), \quad (1.8)$$

where $\mathscr{C}_n^{(\alpha)}(x)$ is a Gegenbauer polynomial [8]. From the solution in Eq. (1.8) one can obtain closed-form expressions for spatial and temporal moments of the random walk [2].

In this paper we generalize the difference equation (1.2) to include the possibility of creation and annihilation of random walkers. We allow random walkers to give birth in region 1 with birth rate a and to die in all other regions with uniform death rate z. Birth rates and death rates are properties of populations rather than of single individuals. Thus, rather than solving Eq. (1.2) as an initial-value problem for a *single* random walker who starts in region m, we are going to study a large population of random walkers, all of whom obey this difference equation. We represent this population of random walkers by a distribution $G_{n,t}$, which denotes the number of random walkers in region n at time t. The distribution $G_{n,t}$ satisfies the same recursion relation as $C_{n,t;m}$ except for the factors of a and z:

$$G_{n,t}=\begin{cases} zP_{\rm in}(n+1)G_{n+1,t-1}+zP_{\rm out}(n-1)G_{n-1,t-1} & (n\geqslant 3)\\ zP_{\rm in}(3)G_{3,t-1}+aG_{1,t-1} & (n=2)\\ zP_{\rm in}(2)G_{2,t-1} & (n=1)\end{cases} \quad (1.9)$$

where we have set $P_{\rm out}(1)=1$. Note that the function $G_{n,t}$ must be positive for all n and t. Aside from the requirement that $G_{n,0}$, the initial distribution of random walkers, be normalizable it is arbitrary. We are not concerned with the detailed structure of the initial distribution; rather, we are interested in the asymptotic behavior of distributions as $t\to\infty$. The specific choice of the initial distribution is unimportant because a random walk is a diffusive (dissipative) process and details of $G_{n,0}$ are irretrievably lost as time evolves; all initial distributions lead to the same large-time behavior. This behavior is determined by the details of the random walk process itself [9].

Differential equations similar to the difference equations in Eq. (1.9) arise in queuing theory [10]. In the context of Ref. [10] the dependent variable $G_{n,t}$ refers to the relative probability of n discrete objects existing at a continuous time t, rather than to a population of random walkers at a discrete site n and discrete time t as in our model. A crucial similarity is that birth and death rates can in general change the total number of walkers from time step to time step in our model, while birth and death rates can in general change the expectation value of the number of objects in queuing theory. Both models are naturally described by three-term recursion relations. Hence the solution techniques employed in this series of papers are similar to those used in Ref. [10] and references therein.

In our model random walkers are created or destroyed at a given site in *proportion* to the number of walkers at that site, where a and z are the constants of proportionality. Technically speaking, a acts as a birth rate if $a>1$; if $a<1$, it is really a death rate. A similar interpretation applies to z. We are particularly interested in steady-state solutions Eq. (1.9); the existence of such solutions imposes a relationship between the birth rate and the death rate.

In Sec. II we perform numerical and analytical studies of Eq. (1.9) for arbitrary choice of $P_{\rm out}(n)$ and $P_{\rm out}(n)$. We study two quantities: N_t, the total number of random walkers at time t, and F_t, the fraction of random walkers in region 1 at time t. We show that the (physically relevant) positive quadrant of the (a,z) plane is partitioned by two intersecting critical curves into four regions (or phases) characterized by the behavior of N_t and F_t as $t\to\infty$. (Some detailed asymptotic studies of the large-t behavior are given in Appendix B.) The intersection of these two critical curves is the critical point (a_c,z_c) for steady-state distributions of random walkers. Although the location of this critical point is a function of the dimension D, the qualitative features of this phase diagram are generic.

We obtain a solvable model of random walks for arbitrary dimension D by using the uniform approximations for $P_{\rm out}(n)$ and $P_{\rm in}(n)$ proposed in Ref. [2] and given in Eq. (1.7). In Sec. III we analyze this model for the simple case $D=1$. We analytically determine the features of the (a,z) phase diagram and verify that random-walk distributions exhibit critical behavior. Then, in Sec. IV we use generating-function methods to solve Eq. (1.9) for arbitrary D. We cannot perform a global analysis of the (a,z) phase diagram as in the case $D=1$, but we can perform a local analysis in the vicinity of the critical point. From this local analysis we show that a second-order phase transition occurs at the critical point.

Specifically, we show that near the critical birth rate a_c, the steady-state distribution fraction $F(a)=\lim_{t\to\infty}F_t$ behaves like

$$F(a)\sim C(D)(a-a_c)^{\nu}\quad (a\to a_c^+,\ D\neq 2,4), \tag{1.10}$$

where the multiplicative constant $C(D)$ depends on the dimension D. The critical exponent ν also depends on the dimension D:

$$\nu=\begin{cases}\dfrac{D}{2-D} & (0<D<2),\\[2ex] \dfrac{2}{D-2} & (2<D<4),\\[2ex] 1 & (D>4).\end{cases} \tag{1.11}$$

There is no critical exponent for the special cases $D=2,4$; instead, we find that as $a\to a_c^+$,

$$F(a)\sim\begin{cases}\dfrac{\rm const}{a-a_c}\,e^{-2/(a-a_c)} & (D=2)\\[3ex] \dfrac{a-a_c}{12\ln\left(\dfrac{1}{a-a_c}\right)} & (D=4).\end{cases} \tag{1.12}$$

In general, formulating simplified D-dimensional statistical models is useful for understanding aspects of critical phenomena exhibited actual physical systems. Indeed, any solvable statistical model that exhibits nontrivial critical behavior is worthy of study [11]. In the next paper in this series we apply the results of this paper to the study of polymer growth in D dimensions. There we extend to arbitrary dimension the earlier results for $D=1$ [12] and $D=2$ [13,14].

II. RANDOM WALKS WITH BIRTH AND DEATH

In this section we discuss the general properties of the spherically symmetric random-walk model defined by Eq. (1.9) in which random walkers may be created and annihilated. Let

$$N_t=\sum_{n=1}^{\infty}G_{n,t} \tag{2.1}$$

be the total number of random walkers at time t. We restrict our attention to initial distributions for which N_0 is finite so that N_t is finite for all t. Let

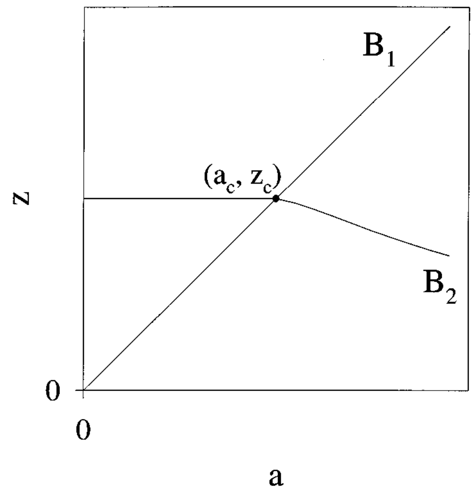

FIG. 1. Generic phase diagram for the (a,z) plane. (The diagram was actually generated using data from $D=\frac{1}{2}$ random walks.) Shown on the diagram are the boundary curves B_1 and B_2. To the left of B_1 and on B_1 the fraction of random walkers in region 1, F_t, approaches 0 as $t\to\infty$; to the right of B_1 this fraction approaches a finite positive number as $t\to\infty$. Above B_2 the total number of random walkers N_t diverges as $t\to\infty$; below B_2 the total number of walkers approaches 0 as $t\to\infty$. On B_2 the distribution of random walkers approaches a steady state as $t\to\infty$. The critical point (a_c,z_c) lies at the intersection of B_1 and B_2.

$$F_t=G_{1,t}/N_t \tag{2.2}$$

represent the *fraction* of all random walkers in region 1 at time t.

Numerical [15] and analytical studies of the quantities N_t and F_t as $t\to\infty$ reveal that, independent of the initial distribution of random walkers, the asymptotic behaviors of N_t and F_t are determined by the values of a and z. Specifically, we obtain the generic result that for any value of D the positive quadrant of the (a,z) plane is partitioned into four distinct regions by two boundary curves as shown in Fig. 1.

One of the boundary curves, which we have labeled B_1 in Fig. 1, is a straight line passing through the origin. To the left of B_1 we find that F_t vanishes as $t\to\infty$; to the right of B_1 we find that F_t approaches a positive finite value as $t\to\infty$. For $D\leqslant 2$ the equation for the boundary line B_1 is $z=a$; as D increases beyond 2 the boundary line remains straight, but the slope of B_1 begins to decrease with increasing D. As we will see, the transition that occurs at $D=2$ is a reflection of Polya's theorem [16], which states that when $D>2$ the probability of an individual random walker visiting region 1 more than once is less than unity.

The second boundary curve shown in Fig. 1 is labeled B_2. This curve consists of two parts: The first part is a straight-line segment $z=1$ extending from the z axis to the boundary line B_1. This line segment connects to the second part, which is a curve that approaches $z=0$ as $a\to\infty$. The equation describing the second part depends on D. [For $D=1$ this curve is given by $z=2a/(a^2+1)(a\geqslant 1)$, as we will show in Sec. III.] Above the boundary B_2 we find that $N_t\to\infty$ as $t\to\infty$; below B_2 we find that $N_t\to 0$ as $t\to\infty$ [17]. On B_2 the total number of walkers approaches a finite distribution $N(a)$ as $t\to\infty$. On the curved portion of B_2 the function $N(a)$ is positive; on the straight-line portion of B_2 the function $N(a)$ is positive for $D>2$, while $N(a)=0$ for $D\leqslant 2$. This transition at $D=2$ is yet another manifestation of Polya's theorem.

While detailed studies of the large-t asymptotic behavior of N_t and F_t are given in Appendix B, many of the qualitative features of Fig. 1 can be derived directly from an analysis of Eq. (1.9). To determine the boundary line B_1 we introduce a change of variable in Eq. (1.9):

$$G_{n,t}=z^t H_{n,t}. \tag{2.3}$$

The distribution $H_{n,t}$ satisfies the recursion relation for a D-dimensional spherically symmetric random walk with a birth rate a/z in region 1 and no births or deaths occurring in any other region:

$$H_{n,t} = \begin{cases} P_{\rm in}(n+1)H_{n+1,t-1}+P_{\rm out}(n-1)H_{n-1,t-1} & (n\geqslant 3) \\ P_{\rm in}(3)H_{3,t-1}+\dfrac{a}{z}H_{1,t-1} & (n=2) \\ P_{\rm in}(2)H_{2,t-1} & (n=1). \end{cases} \tag{2.4}$$

Let $\Pi_1(D)$ denote the probability that a random walker in region 1 will eventually return to region 1. Suppose a random walk satisfying Eq. (2.4) begins at $t=0$. Of the $H_{1,0}$ walkers who begin in region 1, only a fraction $\Pi_1(D)$ of them will eventually return to region 1 to give birth to new walkers at the rate a/z. Of these new walkers, again only a fraction $\Pi_1(D)$ of them will return to region 1 to give birth again and so on. Hence, to find the total number of random walkers who are ever born we must sum a geometric series whose geometric ratio is the quantity $a\Pi_1(D)/z$. If this quantity is less than 1, the geometric series converges and the total number of random walkers ever born is finite. As time t increases, the random walkers diffuse away from region 1. Thus the ratio

$$F_t=\frac{G_{1,t}}{\Sigma_{n=1}^{\infty}G_{n,t}}=\frac{H_{1,t}}{\Sigma_{n=1}^{\infty}H_{n,t}}$$

vanishes as $t\to\infty$. On the other hand, if the quantity $a\Pi_1(D)/z$ is greater than 1, both $H_{1,t}$ and $\Sigma_{n=0}^{\infty}H_{n,t}$ diverge at the same rate and the ratio F_t approaches a nonzero limit (that lies between 0 and 1) as $t\to\infty$.

The transition between $F_t\to 0$ and $F_t\to$(finite limit) occurs on the line

$$z=a\Pi_1(D). \tag{2.5}$$

This is the equation of the boundary line B_1. Polya's theorem states that for any random walk $\Pi_1(D)=1$ when $D\leqslant 2$ and $\Pi_1(D)<1$ when $D>2$. This theorem explains the transition in the slope of the line B_1 at $D=2$. In the spherically symmetric random walk model discussed in Ref. [1], where $P_{\rm out}(n)$ and $P_{\rm in}(n)$ are given in Eqs. (1.4)–(1.6), it was shown that

$$\Pi_1(D)=1-1/\zeta(D-1) \quad (D\geqslant 2) \tag{2.6}$$

(ζ is the Riemann Zeta function); in the random walk model discussed in Ref. [2], where $P_{\rm out}(n)$ and $P_{\rm in}(n)$ are given in Eq. (1.7), it was shown that

$$\Pi_1(D)=1/(D-1) \quad (D\geqslant 2). \tag{2.7}$$

Numerical computation confirms the slope of the boundary line B_1 for both models (see Figs. 2–9).

The shape of the curved part of the boundary B_2 in Fig. 1 depends on the dimension D and on the choice of the functions $P_{\rm out}(n)$ and $P_{\rm in}(n)$. It is not universal. However, the straight-line portion of the boundary B_2 is universal and is easy to understand for any D. Points (a,z) such that $a<a_c$ and z is near 1 lie to the left of B_1. Thus F_t, the fraction of random walkers in region 1, becomes vanishingly small as $t\to\infty$. Hence the effect of the birth rate a on the total number of walkers is negligible. The growth or decay of the total

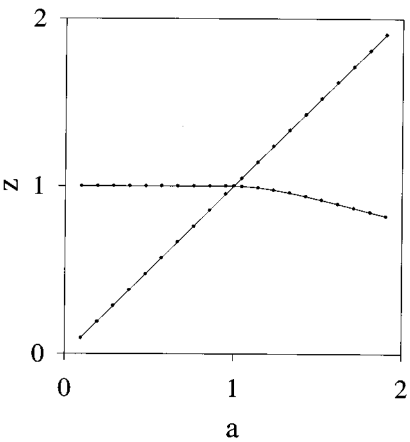

FIG. 2. Phase diagram in the (a,z) plane for the case $D=1$. For this dimension the probabilities $P_{\rm out}(n)$ and $P_{\rm in}(n)$ for the hyperspherical surface area case given in Eqs. (1.4)–(1.6) and the uniform approximation case given in Eqs. (1.7) are the same. On this diagram the slope of B_1 is unity, $a_c=1$, and the slope of B_2 is continuous.

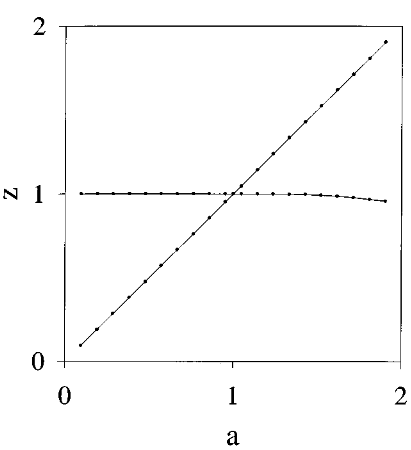

FIG. 3. Phase diagram in the (a,z) plane for the case $D=2$. For this dimension the probabilities $P_{\rm out}(n)$ and $P_{\rm in}(n)$ for the hyperspherical surface area case and the uniform approximation case are the same. On this diagram the slope of B_1 is unity, $a_c=1$, and the slope of B_2 is continuous. Note that the slope of B_2 approaches 0 exponentially fast as $a\to a_c$ from above.

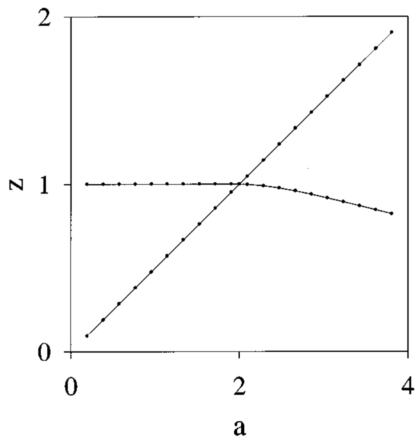

FIG. 4. Phase diagram in the (a,z) plane for the case $D=3$ using the probabilities $P_{\mathrm{out}}(n)$ and $P_{\mathrm{in}}(n)$ for the uniform approximation case in Eqs. (1.7). The slope of B_1 is $\frac{1}{2}$, $a_c=2$, and the slope of B_2 is continuous.

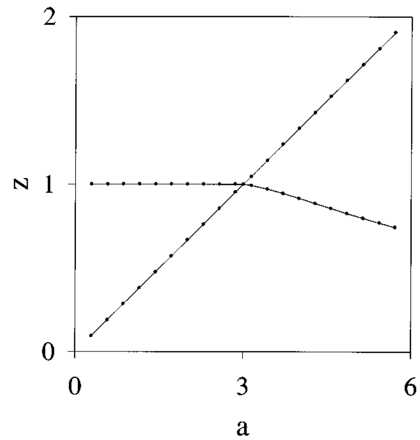

FIG. 6. Phase diagram in the (a,z) plane for the case $D=4$ using the probabilities $P_{\mathrm{out}}(n)$ and $P_{\mathrm{in}}(n)$ for the uniform approximation case given in Eqs. (1.7). The slope of B_1 is $\frac{1}{3}$ and $a_c=3$. The slope of B_2 is continuous; it vanishes logarithmically as $a\rightarrow a_c$ from above [see Eq. (4.21)].

number of walkers only depends on the magnitude of z; if $z<1$ then $N_t\rightarrow 0$ as $t\rightarrow\infty$ and if $z>1$ the $N_t\rightarrow\infty$ as $t\rightarrow\infty$.

On the straight-line portion of the curve B_2, where $a<a_c$ and $z=1$, the limiting value of N_t depends on the dimension D. If $D\leqslant 2$ then $a_c=1$. Thus, on this portion of B_2 a fraction $1-a$ of random walkers who arrive in region 1 at a given time step must die at the next time step. But by Polya's theorem *all* random walkers visit region 1 repeatedly. Hence the total number of random walkers N_t must vanish at $t\rightarrow\infty$. On the other hand, if $D>2$ we have $\Pi_1(D)<1$. Thus the fraction $1-\Pi_1(D)$ of random walkers who originate in region 1 never return to region 1. Thus these random walkers never die because $z=1$. Hence N_t approaches a finite positive number as $t\rightarrow\infty$.

We find numerically that as we cross the boundary line B_1, the limiting value of the function F_t as $t\rightarrow\infty$ is continuous. We are particularly interested in crossing from one side of B_1 to the other along the boundary curve B_2 that divides the upper region, where $N_t\rightarrow\infty$, and the lower region, where $N_t\rightarrow 0$ as $t\rightarrow\infty$. We focus on this curve B_2 because it is only on this curve that a steady state is reached as $t\rightarrow\infty$. Along this boundary curve the limiting value of F_t undergoes a second-order phase transition at the critical point (a_c,z_c), which is situated at the intersection of B_1 and B_2. On the curve B_2 when $a<a_c$ the limiting value of F_t is 0 (even though the limiting value of N_t may be 0) and when $a>a_c$ the limiting values of both N_t and F_t on the boundary curve B_2 are *finite* positive numbers. The curved portion of B_2 is in fact the locus of all points in the positive quadrant of the (a,z) plane for which the limiting values of both N_t and F_t as $t\rightarrow\infty$ are finite and nonzero.

The interpretation of $\lim_{t\rightarrow\infty}N_t$ being finite and nonzero is that the distribution $G_{n,t}$ approaches a steady state. In such a steady-state there is a balance between random walkers being created in region 1 and annihilated in all other regions. This steady-state solution can be obtained by solving a discrete eigenvalue problem.

Steady-state distributions are special cases of shape-

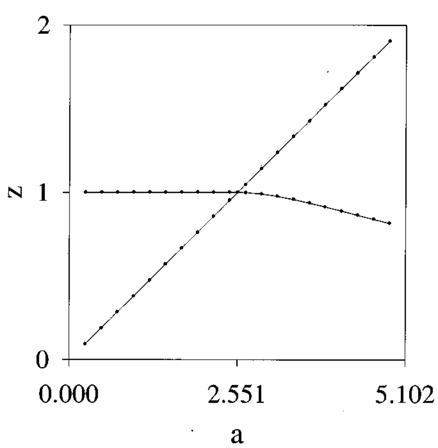

FIG. 5. Phase diagram in the (a,z) plane for the case $D=3$ using the probabilities $P_{\mathrm{out}}(n)$ and $P_{\mathrm{in}}(n)$ for the hyperspherical surface area case given in Eqs. (1.4)–(1.6). The slope of B_1 is $1-1/\zeta(2)=1-6/\pi^2$, $a_c=2.551\ldots$, and the slope of B_2 is continuous.

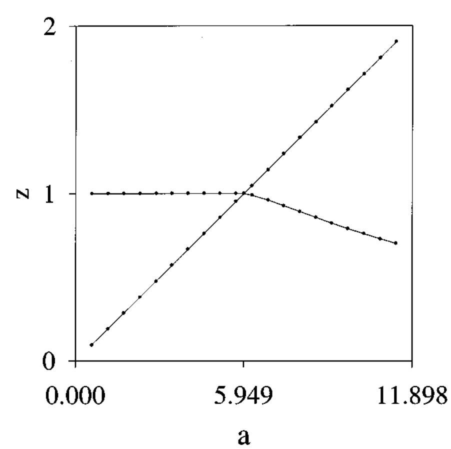

FIG. 7. Phase diagram in the (a,z) plane for the case $D=4$ using the probabilities $P_{\mathrm{out}}(n)$ and $P_{\mathrm{in}}(n)$ for the hyperspherical surface area case given in Eqs. (1.4)–(1.6). The slope of B_1 is $1-1/\zeta(3)$ and $a_c=5.949\ldots$. Universality arguments lead us to believe that the slope of B_2 is continuous and vanishes logarithmically as $a\rightarrow a_c$ from above, as in Fig. 6.

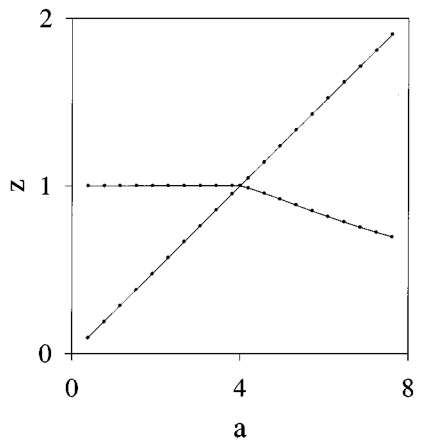

FIG. 8. Phase diagram in the (a,z) plane for the case $D=5$ using the probabilities $P_{\rm out}(n)$ and $P_{\rm in}(n)$ for the uniform approximation case given in Eqs. (1.7). The slope of B_1 is $\frac{1}{4}$ and $a_c=4$. The slope of B_2 is not continuous; there is an elbow at $a=a_c$.

independent distributions; that is, distributions that do not change shape as they evolve in time. For such distributions $G_{n,t}/G_{m,t}$ is independent of t for all n and m so that the relative number of walkers in region n is a time-independent fraction of the total number of walkers. The time dependence of such distributions is very simple:

$$G_{n,t}=g_n\lambda^t. \tag{2.8}$$

The distribution g_n satisfies the discrete eigenvalue problem

$$\lambda g_n=\begin{cases} P_{\rm in}(n+1)zg_{n+1}+P_{\rm out}(n-1)zg_{n-1} & (n\geqslant 3)\\ P_{\rm in}(3)zg_3+ag_1 & (n=2),\\ P_{\rm in}(2)zg_2 & (n=1)\end{cases} \tag{2.9}$$

which is obtained by substituting $G_{n,t}$ in Eq. (2.8) into Eq. (1.9). Here the eigenvalue λ represents the multiplicative growth or decay of the total number of walkers that occurs at each time step. Since we are interested in distributions of random walkers for which the birth rate balances the death rate (so that the total number of walkers is constant in time), we must set $\lambda=1$ in Eq. (2.9). We solve this eigenvalue equation for the case $D=1$ in Sec. III and for the case of arbitrary D in Sec. IV.

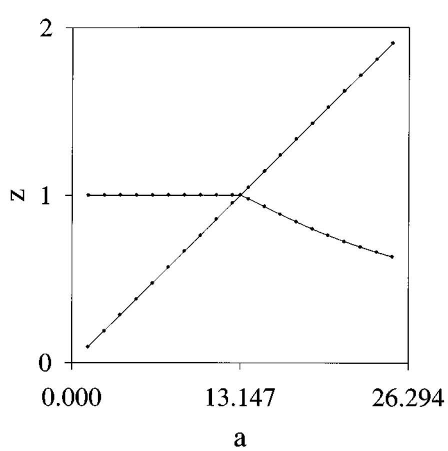

FIG. 9. Phase diagram in the (a,z) plane for the case $D=5$ using the probabilities $P_{\rm out}(n)$ and $P_{\rm in}(n)$ for the hyperspherical surface area case given in Eqs. (1.4)–(1.6). The slope of B_1 is $1-1/\zeta(4)=1-90/\pi^4$ and $a_c=13.147\ldots$. The slope of B_2 is not continuous; there is an elbow at $a=a_c$.

III. ONE-DIMENSIONAL RANDOM WALKS WITH BIRTH AND DEATH

In this section we consider the one-dimensional $(D=1)$ version of the discrete eigenvalue problem Eq. (2.9). When $D=1$, Eqs. (1.4)–(1.6) and (1.7) reduce to

$$P_{\rm out}(n)=\begin{cases}\frac{1}{2} & (n\geqslant 2)\\ 1 & (n=1)\end{cases}$$

and

$$P_{\rm in}(n)=\begin{cases}\frac{1}{2} & (n\geqslant 2)\\ 0 & (n=1).\end{cases}$$

For this case the steady-state distribution obtained by setting $\lambda=1$ in Eq. (2.9) satisfies

$$g_n=\begin{cases}\frac{1}{2}zg_{n+1}+\frac{1}{2}zg_{n-1} & (n\geqslant 3)\\ \frac{1}{2}zg_3+ag_1 & (n=2)\\ \frac{1}{2}zg_2 & (n=1).\end{cases} \tag{3.1}$$

It is easy to solve the difference equation (3.1) because it is a linear constant-coefficient equation. Its general solution has the form

$$g_n=Ar_-^{n-2}+Br_+^{n-2} \quad (n\geqslant 2), \tag{3.2}$$

where

$$r_\pm^2-\frac{2}{z}r_\pm+1=0 \tag{3.3}$$

and A and B are arbitrary constants. The solutions to the quadratic equation (3.3) are

$$r_\pm=\frac{1}{z}\left(1\pm\sqrt{1-z^2}\right). \tag{3.4}$$

Observe that

$$r_-r_+=1. \tag{3.5}$$

Since the total number of random walkers is finite, the sum $\Sigma_{n=1}^{\infty}g_n$ exists. From the existence of this sum and Eq. (3.5) we may conclude that $r_\pm$ are real; if $r_\pm$ were complex then, since they are complex conjugates, we would have $|r_\pm|=1$ and the sum would diverge. Furthermore, since $r_+>1$, it follows that $B=0$.

If we substitute the solution (3.2) with $B=0$ into the special cases ($n=1$ and 2) of Eq. (3.1), we obtain a relationship between the birth rate a and the death rate z:

$$a=r_+=\frac{1}{z}\left(1+\sqrt{1-z^2}\right) \quad \text{or} \quad z=\frac{2a}{a^2+1}. \tag{3.6}$$

This is the equation for the curved part of B_2, the boundary curve between the upper region where $N_t \to \infty$ and the lower region where $N_t \to 0$ when $a \geqslant 1$ [18].

As a function of the birth rate a, the fraction $F(a)$ of random walkers in region 1 for the steady-state distribution g_n is given by

$$F(a) = \frac{g_1}{\sum_{n=1}^{\infty} g_n} = \frac{a-1}{a(a+1)}. \tag{3.7}$$

(Note that the overall multiplicative constant A drops out from this result and is unimportant.) Equation (3.7) is only valid for $a > 1$; if $a \leqslant 1$ then no nontrivial steady-state solution exists; the limiting value of F_t as $t \to \infty$ is 0. Indeed, it is shown in Appendix B that as $t \to \infty$ the fraction F_t vanishes like $1/\sqrt{t}$ along the line B_1 and like $1/t$ everywhere to the left of B_1.

We observe a second-order phase transition in $F(a) = \lim_{t\to\infty} F_t(a)$ as a function of the birth rate a; below the critical birth rate $a_c = 1$ this fraction vanishes and just above the critical point the fraction rises linearly with slope $\frac{1}{2}$:

$$F(a) \sim \tfrac{1}{2}(a - a_c) \quad (a \to 1+). \tag{3.8}$$

Hence, at $D=1$ the critical exponent ν in Eq. (1.10) is 1 and the constant $C(1) = \frac{1}{2}$.

IV. *D*-DIMENSIONAL RANDOM WALKS WITH BIRTH AND DEATH

In this section we generalize the analysis of the preceding section to arbitrary dimension D. When $D \neq 1$ the difference equation (2.9) is no longer a constant-coefficient difference equation and it cannot be solved in closed form. Hence we use the method of generating functions to study steady-state ($\lambda = 1$) solutions of this difference equation.

We seek a solution to the D-dimensional generalization of Eq. (3.1)

$$g_n = \begin{cases} \dfrac{n}{2n+D-1} z g_{n+1} + \dfrac{n+D-3}{2n+D-5} z g_{n-1} & (n \geqslant 3) \\[2mm] \dfrac{2}{D+3} z g_3 + a g_1 & (n=2) \\[2mm] \dfrac{1}{D+1} z g_2 & (n=1), \end{cases} \tag{4.1}$$

which is obtained by substituting the uniform approximations to $P_{\text{out}}(n)$ and $P_{\text{in}}(n)$ given in Eq. (1.7) into Eq. (2.9) and setting $\lambda = 1$.

For a steady-state solution having a finite number of random walkers the sum $\Sigma_{n=1}^{\infty} g_n$ exists. We may therefore sum both sides of Eq. (4.1) from $n=1$ to ∞ and simplify the result:

$$\sum_{n=1}^{\infty} g_n = (a-z) g_1 + z \sum_{n=1}^{\infty} g_n. \tag{4.2}$$

Assuming that the sum $\Sigma_{n=1}^{\infty} g_n$ is nonzero we may immediately conclude that

$$F(a) = \frac{g_1}{\sum_{n=1}^{\infty} g_n} = \frac{1 - z(a)}{a - z(a)}. \tag{4.3}$$

Note that the result in Eq. (4.3) is valid on the curved part of B_2, where the sum exists and is nonzero; it is also valid on the straight-line portion of B_2 when $D > 2$. On the curve B_2 we must treat z as a function of a. We emphasize this dependence by writing $z(a)$ and by treating the fraction F as a function of a only.

In Appendix A we derive an eigenvalue condition from the eigenvalue problem in Eq. (4.1), which yields an implicit equation that determines the curve B_2 in the (a,z) plane:

$$1 = \left(1 - \frac{z}{a}\right) {}_2F_1\left(\frac{1}{2}, 1; \frac{D+1}{2}; z^2\right). \tag{4.4}$$

However, such a higher transcendental equation cannot be solved for z as a function of a in closed form. Thus we perform an asymptotic analysis of this condition for z near 1. [As in the Sec. III, we find that for $z \to 1-$ along B_2 there is a transition at $z=1$ from nontrivial steady-state solutions to trivial solutions of the walk equation (4.1).]

To perform this analysis we let $z = 1 - \eta$. We then use the following formula for the analytic continuation of a hypergeometric function:

$$\begin{aligned} {}_2F_1(a,b;c;\zeta) = {} & \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \\ & \times {}_2F_1(a,b;a+b-c+1;1-\zeta) \\ & + (1-\zeta)^{c-a-b} \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)} \\ & \times {}_2F_1(c-a,c-b;c-a-b+1;1-\zeta). \end{aligned} \tag{4.5}$$

Next, we substitute the first few terms in the Taylor series of a hypergeometric function

$${}_2F_1(a,b;c;\zeta) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{n=0}^{\infty} \frac{\Gamma(a+n)\Gamma(b+n)}{n!\,\Gamma(c+n)} \zeta^n \tag{4.6}$$

to obtain

$$\begin{aligned} 1 \sim {} & \left[\frac{a_c - 1}{a_c} + \frac{a - a_c}{a_c^2} + \frac{\eta}{a_c}\right]\left[\frac{D-1}{D-2}\left(1 + \eta \frac{2}{4-D}\right)\right. \\ & \left. + \eta^{D/2-1} \frac{K}{2-D}\right], \end{aligned} \tag{4.7}$$

where

$$K = \frac{2^{D/2}}{\sqrt{\pi}} \Gamma\left(\frac{D+1}{2}\right) \Gamma\left(2 - \frac{D}{2}\right), \tag{4.8}$$

which is valid near the critical point $(a_c, z_c = 1)$. Note that the value of a_c depends on D and must be determined by Eq. (4.7).

Our results are as follows. Leading-order asymptotic analysis for small η gives the location of the critical point

(a_c,z_c); the critical point lies at (1,1) for $0<D\leqslant 2$ and at $(D-1,1)$ for $D\geqslant 2$. A next-order asymptotic analysis of Eq. (4.7) for the case $0<D<2$ yields

$$z(a)\sim 1-\left(\frac{K}{2-D}\right)^{2/(2-D)}(a-a_c)^{2/(2-D)}\quad (a\to a_c^+) \tag{4.9}$$

and

$$F(a)\sim\left(\frac{K}{2-D}\right)^{2/(2-D)}(a-a_c)^{D/(2-D)}\quad (a\to a_c^+). \tag{4.10}$$

Equation (4.10) reduces to Eq. (3.8) when $D=1$.

For the case $2<D<4$ a next-order asymptotic analysis of Eq. (4.7) gives

$$z(a)\sim 1-[(D-2)K]^{2/(D-2)}(a-a_c)^{2/(D-2)}\quad (a\to a_c^+) \tag{4.11}$$

and

$$F(a)\sim[(D-2)K]^{2/(D-2)}(a-a_c)^{2/(D-2)}\quad (a\to a_c^+). \tag{4.12}$$

When $D>4$ we find that

$$z(a)\sim 1-\frac{D-4}{D(D-1)}(a-a_c)\quad (a\to a_c^+) \tag{4.13}$$

and

$$F(a)\sim\frac{D-4}{D(D-1)(D-2)}(a-a_c)\quad (a\to a_c^+). \tag{4.14}$$

The special case $D=3$ can be solved exactly in closed form

$$z(a)=1-\frac{(a-2)^2}{a^2+4}\quad (a\geqslant a_c=2) \tag{4.15}$$

and

$$F(a)=\frac{(a-2)^2}{a^3}\quad (a\geqslant a_{rmc}=2). \tag{4.16}$$

Indeed, the difference equation (4.1) can be solved exactly and in closed form for *all* odd-integer D; the solution that vanishes as $n\to\infty$ is given by

$$g_n=r_-^n\mathscr{P}_{(D-1)/2}(n),$$

where $\mathscr{P}_k(n)$ is a polynomial in the variable n of degree k. When D is an odd integer the hypergeometric series in Eq. (A14) truncates for $D\geqslant 5$. Unfortunately, except for the cases $D=1$ and 3 we do not obtain a *simple* form for the solution for $z(a)$ and $F(a)$. An implicit solution for $z(a)$ when $D=5$, for example, is given by

$$(9a^2+64)z^3-56az^2-(8a^2+48)z+48a=0.$$

The special cases $D=0$, 2, and 4 need to be treated separately. For $D=0$ the eigenvalue condition Eq. (A14) becomes very simple because we can use the identity

$${}_2F_1(a,b;a;\zeta)=(1-\zeta)^{-b}.$$

Elementary algebra then yields

$$z(a)=\frac{1}{a} \tag{4.17}$$

for all a. Thus the boundary B_2 is a hyperbola for all a; the straight-line portion of B_2 for $a<1$ disappears. To understand this result observe that when $D=0$, Eq. (4.1) states that random walkers in region 2 cannot move outward. The appearance of this restriction is an artifact of the uniform approximation in Eq. (1.7). Thus a steady-state solution has $g_n=0$ for $n>2$ and consists of random walkers oscillating between region 1 and region 2. In this case, the fraction $F(a)$ of walkers in region 1 is exactly

$$F(a)=\frac{1}{1+a}. \tag{4.18}$$

For this degenerate case there is no critical point and no phase transition. We emphasize that the disappearance of a phase transition is an artifact; the uniform approximation in Eq. (1.7) is only valid when $D>0$.

For $D=2$ we find that

$$z(a)\sim 1-\text{const}\ e^{-2/(a-a_c)}\quad (a\to a_c^+) \tag{4.19}$$

and

$$F(a)\sim\frac{\text{const}}{a-a_c}e^{-2/(a-a_c)}\quad (a\to a_c^+). \tag{4.20}$$

For $D=4$ we have

$$z(a)\sim 1-\frac{a-a_c}{6\ln\left(\frac{1}{a-a_c}\right)}\quad (a\to a_c^+) \tag{4.21}$$

and

$$F(a)\sim\frac{a-a_c}{12\ln\left(\frac{1}{a-a_c}\right)}\quad (a\to a_c^+). \tag{4.22}$$

The results in Eqs. (4.9)–(4.22) confirm the formulas given in Eqs. (1.10)–(1.12). Numerical calculations verify the universality of the scaling coefficients given here [3].

The limiting case $D\to\infty$ is interesting because, like the case $D=1$, we can find the exact equation for the curved portion of B_2. To treat this case we perform a large-D asymptotic expansion of the integral in the eigenvalue condition given in Eq. (A13). Using Laplace's method we obtain an asymptotic expansion of this condition as a formal series in powers of $1/D$. We recover from this condition an expression for z as a function of a/a_c:

$$z(a)\sim\frac{a_c}{a}+\frac{2}{D}\left[\frac{a_c}{a}-\left(\frac{a_c}{a}\right)^3\right]+\mathscr{O}(D^{-2})\quad (D\to\infty). \tag{4.23}$$

As one can see from Eq. (4.14), in the limit $D\to\infty$ the transition at $a=a_c=D-1$ is still second order. However, in this limit the discontinuity in the slope of $F(a)$ disappears and $F(a)\to 0$ for all a.

Equations (4.11) and (4.13) indicate that there is a change in the form of the transition at $D=4$. When $D<4$ the slope of the boundary curve B_2 is continuous and the critical exponent depends on D. However, when $D>4$ an elbow appears in B_2 at the critical value $a_c=D-1$ and the critical exponent is independent of D. Specifically, when $D>4$ the slope of B_2 is 0 for $0\leqslant a<D-1$; just above $a=D-1$ the slope abruptly becomes $-(D-4)/D(D-1)$.

We conclude this section by presenting a quick heuristic argument that reproduces the results in Eqs. (4.13) and (4.14). For the case $D>4$ we showed in Ref. [2] that $T_1(D)$, the expected time for a random walker who originates in region 1 to return to region 1, is given by

$$T_1(D)=2\frac{D-2}{D-4}. \tag{4.24}$$

In a steady state all g_1 random walkers in region 1 leave this region and in a $z=1$ model only the fraction $\Pi_1(D)$ ever return. The random walkers who return to region 1 do so in $T_1(D)$ steps on average. These returning random walkers experience a death rate z for $T_1(D)-1$ of these $T_1(D)$ steps. Thus the expected number of random walkers who actually return to region 1 is decreased by the factor $z^{T_1(D)-1}$. Hence, after $T_1(D)$ steps we expect to find $a\Pi_1(D)z^{T_1(D)-1}g_1$ random walkers in region 1. The condition that there be a steady state is therefore given by

$$a\Pi_1(D)z^{T_1(D)-1}=1. \tag{4.25}$$

Using the expressions for $\Pi_1(D)$ and $T_1(D)$ in Eqs. (2.7) and (4.24), we obtain an approximate relation between z and a that is valid near the critical point; that is, where $a=D-1+\delta$, $z=1-\epsilon$ as $\delta,\epsilon\to 0+$. To first order in δ and ϵ this approximate relation is

$$\epsilon\sim\delta\frac{D-4}{D(D-1)}, \tag{4.26}$$

which is precisely the result in Eq. (4.13). We obtain the result in Eq. (4.14) by substituting Eq. (4.13) into Eq. (4.3). Note that this argument is valid only for $a\geqslant a_c=D-1$.

While the above argument is only valid in the neighborhood of a_c, we can also use the above reasoning to derive the entire curve $z(a)$ in the limit $D\to\infty$. In this limit, $T_1=2$. Hence, from Eq. (4.25) we have

$$z=\frac{a_c}{a}, \tag{4.27}$$

the leading behavior in Eq. (4.23).

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APPENDIX A: DERIVATION OF THE EIGENVALUE CONDITION

To obtain the dependence of z on a along B_2 in the eigenvalue condition Eq. (4.4), we use generating function methods. To begin, we simplify Eq. (4.1) by setting

$$g_n=(2n+D-3)h_n. \tag{A1}$$

Substituting Eq. (A1) into Eq. (4.1) we obtain

$$(2n+D-3)h_n=\begin{cases} nzh_{n+1}+(n+D-3)zh_{n-1} & (n\geqslant 3)\\ 2zh_3+(D-1)ah_1 & (n=2)\\ zh_2 & (n=1).\end{cases} \tag{A2}$$

Next, we define a generating function

$$H(x)=\sum_{n=0}^{\infty}x^nh_{n+1}. \tag{A3}$$

Note that

$$G(x)=\sum_{n=0}^{\infty}x^ng_{n+1}=\left(2x\frac{d}{dx}+D-1\right)H(x). \tag{A4}$$

Multiplying Eq. (A2) by x^{n-1} and summing both sides from $n=3$ to ∞, we obtain a first-order inhomogeneous linear differential equation for $H(x)$:

$$(zx^2-2x+z)H'(x)+(D-1)(zx-1)H(x)=(z-a)(D-1)xh_1. \tag{A5}$$

To solve Eq. (A5) we multiply both sides by the integrating factor $(x^2z-2x+z)^{(D-3)/2}$. The differential equation then simplifies to

$$\frac{d}{dx}[(zx^2-2x+z)^{(D-1)/2}H(x)]=(z-a)g_1x(x^2z-2x+z)^{(D-3)/2}. \tag{A6}$$

The general solution to Eq. (A6) is

$$H(x)=(zx^2-2x+z)^{(1-D)/2}\left[C+(z-a)g_1\int_0^x ds\ s(s^2z-2s+z)^{(D-3)/2}\right], \tag{A7}$$

where C is an arbitrary constant.

To determine the constant C we observe that $H(0)=h_1=g_1/(D-1)$, from which it follows that

$$C=\frac{g_1}{D-1}\,z^{(D-1)/2}.$$

Thus

$$H(x)=g_1\left(x^2-2\frac{x}{z}+1\right)^{(1-D)/2}\left[\frac{1}{D-1}+\left(1-\frac{a}{z}\right)\times\int_0^x ds\ s\left(s^2-2\frac{s}{z}+1\right)^{(D-3)/2}\right]. \tag{A8}$$

Finally, we use Eq. (A4) to obtain the generating function $G(x)$:

$$G(x)=g_1\left\{\left(x^2-2\frac{x}{z}+1\right)^{-(D+1)/2}(1-x^2)\left[1+(D-1)\times\left(1-\frac{a}{z}\right)\int_0^x ds\ s\left(s^2-2\frac{s}{z}+1\right)^{(D-3)/2}\right]+\frac{2x^2\left(1-\frac{a}{z}\right)}{x^2-2\frac{x}{z}+1}\right\}. \tag{A9}$$

Assuming that $G(x)$ exists for all $0\leqslant x\leqslant 1$, we formally recover Eq. (4.3) when we set $x=1$.

Recall the quantities $r_\pm$ defined in Eq. (3.4) and rewrite Eq. (A9) as

$$G(x)=g_1\left\{[(r_+-x)(r_--x)]^{-(D+1)/2}(1-x^2)\left[1+(D-1)\times\left(1-\frac{a}{z}\right)\int_0^x ds\ s(r_+-s)^{(D-3)/2}(r_--s)^{(D-3)/2}\right]+\frac{2x^2\left(1-\frac{a}{z}\right)}{(r_+-x)(r_--x)}\right\}. \tag{A10}$$

For $z<1$ the generating function $G(x)$ may be singular at $x=r_-<1$, in which case the representation of $G(1)$ as a series will not exist. To preclude the possibility of such a singularity it is necessary and sufficient to impose the eigenvalue condition

$$1+(D-1)\left(1-\frac{a}{z}\right)\int_0^{r_-} ds\ s(r_+-s)^{(D-3)/2}(r_--s)^{(D-3)/2}=0. \tag{A11}$$

This condition is clearly necessary. We can verify that it is sufficient by showing that $G(r_--\epsilon)$ exists in the limit as $\epsilon\to 0+$. To leading order in ϵ the eigenvalue condition in Eq. (A11) becomes

$$1+(D-1)\left(1-\frac{a}{z}\right)\int_0^{r_--\epsilon} ds\, s(r_+-s)^{(D-3)/2}(r_--s)^{(D-3)/2}\sim -2\left(1-\frac{a}{z}\right)r_-(r_+-r_-)^{(D-3)/2}\epsilon^{(D-1)/2}\quad(\epsilon\to 0+). \tag{A12}$$

Substituting this asymptotic result into Eq. (A10), we see that the last term, which is of order ϵ^{-1}, exactly cancels.

The eigenvalue condition in Eq. (A11) expresses the relation between a and z that we seek. We can rewrite this condition more compactly by rescaling the integration variable. Let $s=r_-u$, so that

$$(D-1)r_-^2\left(\frac{a}{z}-1\right)\int_0^1 du\ u(1-u)^{(D-3)/2}(1-r_-^2u)^{(D-3)/2}=1. \tag{A13}$$

This integral converges only if $D>1$. We can analytically continue to values $0<D\leqslant 1$ [recall that the region of validity of the uniform approximation in Eq. (1.7) is $D>0$] by recognizing that this expression contains the standard integral representation for a hypergeometric function [8]

$$\frac{4r_-^2}{D+1}\left(\frac{a}{z}-1\right)\ {}_2F_1\left(\frac{3-D}{2},2;\frac{D+3}{2};r_-^2\right)=1. \tag{A14}$$

Using the transformation formulas (especially 15.3.26, 15.2.18, and 15.2.20 in Ref. [8]) for hypergeometric functions, this form of the eigenvalue condition can be simplified to obtain Eq. (4.4).

APPENDIX B: LARGE-TIME ASYMPTOTIC BEHAVIOR

In this appendix we analyze the large-t behavior of the distribution $G_{n,t}$ of random walkers for the uniform approximation of the probabilities $P_{\rm out}$ and $P_{\rm in}$ given in Eq. (1.7). To this end we solve the set of equations in (1.9) for the Kronecker delta initial condition $G_{n,0}=\delta_{n,1}$. As discussed earlier, the large-t behavior of a dissipative process is independent of the specific choice of initial condition.

First, we derive a formal solution for $G_{n,t}$ that is valid for general $P_{\rm out}$ and $P_{\rm in}$. We define

$$d_{n,t}=\begin{cases} az^{n-2}\left(\prod_{i=1}^{n-1}P_{\rm out}(i)\right)G_{n,t} & (n\geqslant 2)\\ G_{1,t} & (n=1)\end{cases} \tag{B1}$$

and rewrite Eqs. (1.9) as

$$d_{n,t}=\begin{cases} Q_nd_{n+1,t-1}+d_{n-1,t-1} & (n\geqslant 2)\\ Q_1d_{2,t-1} & (n=1),\end{cases} \tag{B2}$$

where we let

$$Q_n=\begin{cases} z^2P_{\rm out}(n)P_{\rm in}(n+1) & (n\geqslant 2)\\ azP_{\rm out}(1)P_{\rm in}(2) & (n=1).\end{cases} \tag{B3}$$

Next, we define the generating function

$$e_n(y)=\sum_{t=0}^{\infty} d_{n,t}y^t \tag{B4}$$

and obtain from Eqs. (B2)

$$e_n(y)=\begin{cases} yQ_n e_{n+1}(y)+ye_{n-1}(y) & (n\geqslant 2)\\ 1+yQ_1e_2(y) & (n=1),\end{cases} \tag{B5}$$

where we have applied the Kronecker delta initial condition.

Let us define a continued fraction by the recursion relation

$$S_n(y^2)=\frac{1}{1-y^2Q_nS_{n+1}(y^2)} \quad (n\geqslant 1). \tag{B6}$$

It is easy to show that for $n\geqslant 3$ the recursion relation in Eq. (B5) is satisfied by

$$e_n(y)=Ay^{n-1}\prod_{i=2}^{n} S_i(y^2) \quad (n\geqslant 2). \tag{B7}$$

[Since $e_n(y)$ obeys a second-order difference equation, there is a linearly independent solution that can be determined using the technique of variation of parameters. This solution does not contribute; apparently, it fails to obey the appropriate boundary conditions at $n=\infty$.] We determine e_1 and the constant A by solving simultaneously the special cases $n=1$ and 2 of Eqs. (B5):

$$\begin{gathered} AyS_2(y^2)=AQ_2y^3S_2(y^2)S_3(y^2)+ye_1(y),\\ e_1(y)=1+Ay^2Q_1S_2(y^2).\end{gathered} \tag{B8}$$

Solving the above equations leads to a surprisingly compact expression for all $e_n(y)$:

$$e_n(y)=y^{n-1}\prod_{i=1}^{n} S_i(y^2) \quad (n\geqslant 1). \tag{B9}$$

Using a contour integral to project out the coefficients in the generating function we obtain

$$G_{n,t}=\left(\frac{a}{z}\right)^{1-\delta_{1,n}}\left[\prod_{i=1}^{n-1} zP_{\rm out}(i)\right]\oint_C \frac{dy}{2\pi iy}\, y^{n-t-1}\prod_{i=1}^{n} S_i(y^2), \tag{B10}$$

where empty products are defined to be unity. The contour C encircles the pole at the origin in the complex-y plane but excludes all other singularities of the integrand.

This rather strange expression (a contour integral over a product of continued fractions) is of little use, even in an asymptotic analysis for large values of t. Only for particular choices for $P_{\rm out}$ and $P_{\rm in}$ is progress possible. A significant advantage of the uniform approximation in Eq. (1.7) [compared with probabilities given in Eqs. (1.4)–(1.6)] is that they simplify the expression for $G_{n,t}$ in Eq. (B10) for all $D>0$. [The probabilities in Eqs. (1.4)–(1.6)] lead to a tractable result only for $D=0$, 1, and 2.]

We simplify the expression for $G_{n,t}$ in Eq. (B10) by recalling the continued-fraction representation for a hypergeometric function [19]:

$$\frac{{}_2F_1(a,b+1;c+1;\zeta)}{{}_2F_1(a,b;c;\zeta)} =1/(1+f_1\zeta/\{1+f_2\zeta/[1+f_3\zeta/(1+\cdots)]\}), \tag{B11}$$

where

$$f_{2i}=-\frac{(i+b)(i+c-a)}{(2i+c)(2i+c-1)},$$

$$f_{2i+1}=-\frac{(i+a)(i+c-a)}{(2i+c)(2i+c+1)}. \tag{B12}$$

Substituting the uniform approximation for $P_{\rm out}$ and $P_{\rm in}$ in Eq. (1.7) into Eq. (B3) gives

$$Q_n=z^2\,\frac{n(n+D-2)}{(2n+D-3)(2n+D+1)} \quad (n\geqslant 2), \tag{B13}$$

which can be rewritten as

$$Q_{n+2i}=z^2\,\frac{\left(i+\frac{n+D-2}{2}\right)\left(i+\frac{n}{2}\right)}{\left(2i+n+\frac{D-3}{2}\right)\left(2i+n+\frac{D-1}{2}\right)} \quad (n\geqslant 2, i\geqslant 1),$$

$$Q_{n+2i+1}=z^2\,\frac{\left(i+\frac{n+D-1}{2}\right)\left(i+\frac{n+1}{2}\right)}{\left(2i+n+\frac{D-1}{2}\right)\left(2i+n+\frac{D+1}{2}\right)} \quad (n\geqslant 2, i\geqslant 0). \tag{B14}$$

The continued fractions in Eq. (B6) can thus be identified as

$$S_n(y^2)=\frac{{}_2F_1\left(\frac{n}{2},\frac{n+1}{2};n+\frac{D-1}{2};z^2y^2\right)}{{}_2F_1\left(\frac{n}{2},\frac{n-1}{2};n+\frac{D-3}{2};z^2y^2\right)} \quad (n\geqslant 2). \tag{B15}$$

Hypergeometric functions are symmetric in their first two arguments. Therefore,

$$\prod_{i=2}^{n} S_i(y^2) = \frac{{}_2F_1\left(\frac{n}{2},\frac{n+1}{2};n+\frac{D-1}{2};z^2y^2\right)}{{}_2F_1\left(\frac{1}{2},1;\frac{D+1}{2};z^2y^2\right)} \quad (n \geqslant 2). \tag{B16}$$

Substituting this last result into Eq. (B10), we finally obtain

$$G_{n,t} = z^t \frac{\Gamma\left(\frac{D+1}{2}\right)\Gamma(n+D-2)}{2^{n-2}\Gamma(D)\Gamma\left(n+\frac{D-3}{2}\right)} \oint_C \frac{dy}{2\pi i y} \times y^{n-t-1} \frac{\left(\frac{z}{a}\right)^{\delta_{1,n}} {}_2F_1\left(\frac{n}{2},\frac{n+1}{2};n+\frac{D-1}{2};y^2\right)}{1+\left(\frac{z}{a}-1\right){}_2F_1\left(\frac{1}{2},1;\frac{D+1}{2};y^2\right)}. \tag{B17}$$

Recalling Eq. (2.1), we obtain an expression for the total number of walkers at time t by summing Eqs. (1.9) over all positive integers n:

$$N_t = z^t\left[1+\left(\frac{a}{z}-1\right)\sum_{\tau=0}^{t-1} z^{-\tau} G_{1,\tau}\right]. \tag{B18}$$

Next we insert $G_{1,t}$ from Eq. (B17) and sum over τ:

$$N_t = z^t\left[1+\left(1-\frac{z}{a}\right)\oint_C \frac{dy\ y^{-t}}{2\pi i(1-y)} \times \frac{{}_2F_1\left(\frac{1}{2},1;\frac{D+1}{2};y^2\right)}{1+\left(\frac{z}{a}-1\right){}_2F_1\left(\frac{1}{2},1;\frac{D+1}{2};y^2\right)}\right], \tag{B19}$$

where we have eliminated terms in the integrand that are regular at the origin in the complex-y plane. From $G_{1,t}$ in Eq. (B17) and N_t in Eq. (B19) we obtain the large-t behavior of the fraction F_t in Eq. (2.2). Note the similarity of the denominator in both integrals with the eigenvalue condition in (4.4) for steady-state solutions. The asymptotic behavior of the integrals for large t is dominated by the poles of the integrands and the steady-state solution is merely the special case where the asymptotic behavior is independent of t in leading order.

To extract the large-t behavior of N_t and F_t, we conduct a saddle-point analysis of the contour integrals for $G_{1,t}$ and N_t in Eqs. (B17) and (B19). Both expressions, aside from the prefactor z^t, only depend on the ratio a/z. Saddle-point analysis requires that we consider three distinct cases: values of a/z such that (i) (a,z) lies to the left of the line B_1, (ii) (a,z) is on B_1, and (iii) (a,z) lies to the right of B_1 (see Fig. 1). For all cases the integrands for both $G_{1,t}$ and N_t have a pole at $y=y_p$ on the real positive axis. For case (i) $y_p>1$, for case (ii) $y_p=1$, and for case (iii) $y_p<1$.

For case (i) we find, to leading order, that as $t\to\infty$

$$G_{1,t} \sim \begin{cases} \dfrac{\frac{a}{z}}{\left(1-\frac{a}{z}\right)^2} \dfrac{(2-D)^2}{2K\Gamma\left(\frac{D}{2}\right)} z^t t^{D/2-2} & (0<D<2) \\[2ex] \dfrac{\frac{a}{(D-1)z}}{\left(1-\frac{a}{(D-1)z}\right)^2} \dfrac{2^{D/2-2}(D-2)^2\Gamma\left(\frac{D-1}{2}\right)}{\sqrt{\pi}} z^t t^{-D/2} & (D>2) \end{cases} \tag{B20}$$

and

$$N_t \sim \begin{cases} \dfrac{\frac{a}{z}}{\left(1-\frac{a}{z}\right)} \dfrac{(2-D)}{K\Gamma\left(\frac{D}{2}\right)} z^t t^{D/2-1} & (0<D<2) \\[2ex] \dfrac{\frac{a}{(D-1)z}}{1-\frac{a}{(D-1)z}} (D-2) z^t & (D>2), \end{cases} \tag{B21}$$

where K is given in Eq. (4.8). Hence

$$F_t \sim \begin{cases} \dfrac{1-\dfrac{D}{2}}{1-\dfrac{a}{z}}\, t^{-1} & (0<D<2) \\[2ex] \dfrac{1}{1-\dfrac{a}{(D-1)z}} \, \dfrac{2^{D/2-2}(D-2)\Gamma\left(\dfrac{D-1}{2}\right)}{\sqrt{\pi}}\, t^{-D/2} & (D>2). \end{cases} \tag{B22}$$

For case (ii) we find, to leading order, that as $t\to\infty$

$$G_{1,t} \sim \begin{cases} \dfrac{K}{2\Gamma\left(2-\dfrac{D}{2}\right)}\, z^t t^{-D/2} & (0<D<2) \\[2ex] \dfrac{D-1}{2K\Gamma\left(\dfrac{D}{2}\right)}\, z^t t^{D/2-2} & (2<D<4) \\[2ex] \dfrac{D-4}{2(D-2)}\, z^t & (D>4) \end{cases} \tag{B23}$$

and

$$N_t \sim \begin{cases} z^t & (0<D<2) \\[1ex] \dfrac{D-1}{K\Gamma\left(\dfrac{D}{2}\right)}\, z^t t^{D/2-1} & (2<D<4) \\[2ex] \dfrac{D-4}{2}\, z^t t & (D>4). \end{cases} \tag{B24}$$

Hence,

$$F_t \sim \begin{cases} \dfrac{D-1}{K\Gamma\left(\dfrac{D}{2}\right)}\, t^{-D/2} & (0<D<2) \\[2ex] \dfrac{1}{2t} & (2<D<4) \\[2ex] \dfrac{1}{(D-2)t} & (D>4). \end{cases} \tag{B25}$$

The analysis of case (iii) is somewhat more complicated. The saddle point in cases (i) and (ii) is very near $y=1$ for large t, but in case (iii) the integrands have poles at $0<y=y_p(a/z)<1$ and the saddle point is now located near y_p. An asymptotic analysis of this case for large t is possible only if we consider a small neighborhood to the right of the line B_1. Approaching B_1 we find that $y_p\to 1-$. We use the Ansätze $a/z=a_c/z_c+\epsilon$ and $y_p(a/z)=1-\delta(\epsilon)$, where $\epsilon\ll 1$ and $\delta\ll 1$, but where $t\delta\gg 1$. We find that as $t\to\infty$

$$\delta(\epsilon) \sim \begin{cases} \epsilon^{2/(2-D)}\left[\dfrac{K}{2-D}\right]^{2/(2-D)} & (0<D<2) \\[2ex] \epsilon^{2/(D-2)}[K(D-2)]^{-2/(D-2)} & (2<D<4) \\[1ex] \epsilon\, \dfrac{D-4}{2(D-1)(D-2)} & (D>4), \end{cases} \tag{B26}$$

leading to

$$G_{1,t}\sim\begin{cases}\epsilon^{D/(2-D)}\,\dfrac{2}{2-D}\left[\dfrac{K}{2-d}\right]^{2/(2-D)}\left[\dfrac{z}{y_p(a/z)}\right]^t & (0<D<2)\\ \epsilon^{(4-D)/(D-2)}\,\dfrac{2(D-1)}{D-2}\,[K(D-2)]^{-2/(D-2)}\left[\dfrac{z}{y_p(a/z)}\right]^t & (2<D<4)\\ \dfrac{D-4}{2(D-2)}\left[\dfrac{z}{y_p(a/z)}\right]^t & (D>4)\end{cases}\tag{B27}$$

and

$$N_t\sim\begin{cases}\dfrac{2}{2-D}\left[\dfrac{z}{y_p(a/z)}\right]^t & (0<D<2)\\ \epsilon^{-1}2(D-1)\left[\dfrac{z}{y_p(a/z)}\right]^t & (2<D<4).\\ \epsilon^{-1}(D-1)(D-2)\left[\dfrac{z}{y_p(a/z)}\right]^t & (D>4).\end{cases}\tag{B28}$$

Hence

$$F_t\sim\begin{cases}\epsilon^{D/(2-D)}\left[\dfrac{K}{2-D}\right]^{2/(2-D)} & (0<D<2)\\ \epsilon^{2/(D-2)}\,\dfrac{[K(D-2)]^{-2/(D-2)}}{D-2} & (2<D<4)\\ \epsilon\,\dfrac{D-4}{2(D-1)(D-2)^2} & (D>4).\end{cases}\tag{B29}$$

From the previous formula we can recover the asymptotic results in Eqs. (4.10), (4.12), and (4.14), which are valid on the line B_2 as $a\to a_c^+$ and $z\to z_c^-$. The particular path B_2 is distinguished merely by the fact that the total number of walkers N_t approaches a nonzero constant as $t\to\infty$. Thus the portion of B_2 to the right of B_1 is obtained for $z=y_p(a/z)$ in Eqs. (B28). Again, we let $z=1-\eta$ for $\eta\to 0+$ and find that $\eta\sim\delta(\epsilon)$. Then, using $z_c=1$, we find that $\epsilon\sim(a-a_c)+a_c\eta$. From Eq. (B26) for $D<4$, we have $\epsilon\gg\eta$ and we merely need to identify $\epsilon=a-a_c$ in Eq. (B29) to recover our earlier results. For $D>4$ we recall that $\epsilon=O(\eta)$ to recover Eq. (4.14).

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[9] We note that C. Koplik *et al.* [Phys. Rev. E **50**, 4650 (1994)] have studied the diffusion of tracer particles in a fluid with a radial velocity field. They observed that the velocity field can be chosen to obtain the continuum limit of a spherically symmetric random walk of particles in another dimension where no velocity field is present. Our discrete random-walk model yields the same differential equations in the continuum limit.

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[15] One must be careful when performing numerical studies of Eq. (1.9). This equation describes random walkers who *must* move

outward or inward at every step. Thus, on a (n,t) lattice the system exhibits a checkerboard parity behavior; it is actually two noninteracting systems, one with sites for which $n+t$ is even and the other with sites where $n+t$ is odd. Hence numerical studies of F_t and N_t with arbitrary initial conditions in general yield oscillatory time evolution. To avoid ambiguities one must focus on just one system. To do so one studies N_t at a given time only for even values of $n+t$ or only for odd values of $n+t$. One must be equally careful about numerical studies of F_t. (In principle, one can couple the two distinct systems by allowing walkers to have a small probability ϵ to *stay* at their current site at each time step. This coupling would enable the systems to thermalize. Once this has happened, one can set $\epsilon=0$ and ignore the distinctions arising from the checkerboard parity.)

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[17] Redner and Kang [4] consider birth and death rates for random walkers in the context of molecular reactions. They generate phase diagrams that have a line like the $B2$ line in Fig. 1, separating growth from decay. However, they consider only integer-dimension hypercubic lattices, have nearest-neighbor steps with equal probability, and *do not* consider the possibility of critical behavior.

[18] D. ben-Avraham, S. Redner, and Z. Cheng (Ref. [5]) consider three one-dimensional geometries on a lattice: (i) a single source or trap, (ii) a source-trap dipole separated by an arbitrary distance, and (iii) a periodic set of sources with traps in between. The traps may be partial traps in all cases, that is, $z\neq 0$ necessarily. They also develop a steady-state balance criterion defining the line B2, the line separating growth and decay, for the second and third geometry. They discuss the limit where the separation between periodic sources goes to infinity. This limit is equivalent to our model in one dimension, since the (partial) traps in between now extend to infinity. Their steady-state condition in this limit is exactly the same as ours for one dimension. Critical behavior is not mentioned.

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